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1987 J. Phys. A: Math. Gen. 20 5351

(http://iopscience.iop.org/0305-4470/20/15/046)

J. Phys. A: Math. Gen. **20** (1987) 5351-5360. Printed in the UK

Conventional zeta-function derivation of high-temperature expansions

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Received 14 January 1987

Abstract. Zeta-function regularisation (of functional determinants) is used to derive the exact high-temperature expansions of the thermodynamic potentials for the ideal massive Bose and Fermi gases for non-zero chemical potential. The purpose is to show agreement with the results obtained by the Mellin transform method, and by the method of zeta-function regularisation of infinite series. The generalised zeta functions calculated are the simplest examples of Feynman diagram zeta functions. The extension of these calculations to Euclidean manifolds $M \times E^n$ more complicated than the finite-temperature cylinder $S^1 \times E^n$ is discussed.

1. Introduction

The thermodynamic potentials for the massive ideal Bose and Fermi gases $-\Omega_{\mathrm{B,F}}(M, \mu)$ where M, μ are mass and chemical potential—are complicated functions [1-7] whose exact power series expansions have only recently been obtained. Haber and Weldon [4, 6] and Braden [7] first derived these series, using a Mellin transformation method. Subsequently, the author [8, 9] calculated the high-T series for $\Omega_{\mathrm{B,F}}(M, \mu)$ in a different way, using ζ functions to rearrange a Dirichlet-type infinite series representing the thermodynamic potential into the high-T power series form. (One distinctive term in Ω_{B}, but none in Ω_{F}, appeared to be missed by this calculation. Weldon [10] pointed out that this term does in fact arise in the ζ-function calculation through the commutation of infinite series. See also [11].) More recently, a slight discrepancy between the Mellin transform result for $\Omega_{\mathrm{F}}(M, \mu)$ [7] and the ζ-function result [9] was resolved in favour of the latter by Landsmann and van Weert [12], who repeated the Mellin transform calculation (see also § 2 below). Thus, there presently exist two quite different calculations of the high-T series for $\Omega_{\mathrm{B,F}}(M, \mu)$, and the results agree perfectly.

One purpose of this article is to give a third derivation of these high-T series for the interesting case of even spacetime dimension. Straightforward ζ-function regularisation of $\ln \det(-D^2 + M^2)$ will be used. Up to a single term, exactly the same double-infinite series is found. An arbitrary scaling constant in the ζ function gets transferred to the numerical coefficient of this one term, which consequently must be 'fit' by comparison with the corresponding term in the known series. Every other term in the latter is precisely reproduced in a simple way. In the relatively trivial [13] case of zero mass, the scaling constant does not enter into the high-T series.

In a sense the true subject of the present paper is the ζ-function method itself, rather than thermodynamic potentials. We calculate $\Omega_{\mathrm{B,F}}$ from the ζ-functions $Z_{\mathrm{B,F}}(s)$

associated with the operator $-D^2+M^2$ on the Euclidean spacetime cylinder $S^1\times E^n$. To be able to do this, we first calculate the high-T expansions of the ζ-functions $Z_{B,F}(s)$. Then, from the prescription [14] $\ln\det(\text{operator}) = -\zeta'(0|\text{operator})$, the high-$T$ expansions of $\Omega_{B,F}$ quickly follow.

The ζ functions $Z_{B,F}(s)$ are also the ζ functions associated with the boson, fermion vacuum loops in $T>0$ Euclidean field theory. In general, a ζ function $\zeta(s|\text{diagram})$ can be associated with *any* Feynman diagram in $T>0$ Euclidean field theory [15]. The prescription is to replace propagators $(k^2+M^2)^{-1}$ by $(k^2+M^2)^{-s}$ around closed loops, where s is complex. The 'Feynman diagram ζ-function' $\zeta(s|\text{diagram})$ provides ultraviolet regularisation of the diagram when needed. Moreover, from the high-T expansion of $\zeta(s|\text{diagram})$, one obtains the high-T series of the diagram by letting $s\to 1$. The calculation of $\Omega_{B,F}$ from $Z_{B,F}(s)$ in the present paper is therefore a specific application of a quite general calculational procedure.

These field theory considerations on the $T>0$ spacetime cylinder $S^1\times E^n$ have a straightforward extension to more complicated spacetime manifolds $M\times E^n$. The new feature encountered is the ζ function associated with the manifold M. The ζ function associated with $M=S^1$ is of course the Riemann ζ function. For the torus $M=S^1\times\ldots\times S^1$ one encounters Epstein ζ functions, and generalisations of these. Other manifolds M with constant curvature will have their own ζ functions $\zeta_M(s)$. One can always express the ζ function of the operator $-D^2+M^2$ on M in terms of $\zeta_M(s)$ and related ζ functions. The problem of calculating $\ln\det(-D^2+M^2)$, and more generally of obtaining Feynman diagram ζ functions on $M\times E^n$, can be solved by first calculating $\zeta_M(s)$ and other ζ functions of this type. In § 3 we sketch this problem without analysing in detail the ζ functions involved, which would take us well beyond the scope of this paper.

2. Euclidean finite-temperature field theory

2.1. Vacuum loop zeta functions

The thermodynamic potentials $\Omega_{B,F}$ for the relativistic Bose and Fermi gases are

$$\beta\Omega_B = \ln\det_+(-D^2+M^2) \tag{2.1}$$

$$\beta\Omega_F = -(d/2)\ln\det_-(-D^2+M^2). \tag{2.2}$$

Here $D_\mu = \partial_\mu - \mathrm{i}(A_0, \mathbf{0})$ and $A_0 = -\mathrm{i}\mu = \text{constant}$ with μ the chemical potential of the boson/fermion gas. (For gauge theories A_0 can meaningfully be given a constant real term as well [9], but we disregard this aspect here.) $\det_\pm$ means the functional determinant is calculated on the space of functions periodic/antiperiodic around the $T>0$ spacetime cylinder $S^1\times E^n$ whose circumference is the inverse temperature $\beta=1/T$. In (2.2), d is the dimension of the Dirac representation for $n+1$ spacetime dimensions.

To calculate the functional determinants (2.1), (2.2) we use the standard ζ-function prescription [14]

$$\ln\det_+(-D^2+M^2) = -Z'_B(0) \tag{2.3}$$

$$\ln\det_-(-D^2+M^2) = -Z'_F(0). \tag{2.4}$$

Here $Z_{B,F}(s)$ are the ζ functions

$$Z_{B,F}(s)(M\,e^{c})^{-2s}$$

$$\equiv \frac{V}{(2\pi)^n}\int d^n k \sum_{m=-\infty}^{\infty} [(\omega_m^{\pm}-A_0)^2+\boldsymbol{k}^2+M^2]^{-s}$$

$$= VT^n\pi^{n/2}\left(\frac{\beta}{2\pi}\right)^{2s}\Gamma(s-n/2)/\Gamma(s)$$

$$\times\begin{cases}\sum_m [(m-u)^2+v^2]^{-s+n/2} & \text{B}\\ \sum_m [(m+\frac{1}{2}-u)^2+v^2]^{-s+n/2} & \text{F.}\end{cases} \tag{2.5}$$

$\omega_m^+=2\pi m/\beta$, $\omega_m^-=2\pi(m+\frac{1}{2})/\beta$ are the allowed energies for bosonic, fermionic quantum fluctuations on the cylinder, and $u=\beta A_0/2\pi$, $v=\beta M/2\pi$ are A_0, M rescaled to be dimensionless parameters. V is the volume of n-dimensional space. The momentum integral in (2.5) is elementary. The factor $(M\exp C)^{2s}$ in (2.5) sets the scale of the ζ functions $Z_{B,F}(s)$, and C is an unspecified dimensionless scale constant. The value of C depends on how regularisation is performed. C appears in a single term in the thermodynamic potentials $\Omega_{B,F}$. We shall fit C to the high-T expansions of $\Omega_{B,F}$ previously derived.

The evaluation of the Matsubara sums in (2.5) is not difficult. Some properties of these sums have been obtained previously [16-18]. However, we require the full dependence on s, u, v. In the boson case this can be conveniently displayed by the following power series in u, v:

$$\sum_{m=-\infty}^{\infty}[(m-u)^2+v^2]^{-s+n/2}=(u^2+v^2)^{-s+n/2}+Z(s-n/2,u,v)+Z(s-n/2,-u,v) \tag{2.6}$$

where, using the binomial expansion twice,

$$Z(s-n/2,u,v)\equiv\sum_{m=1}^{\infty}[(m+u)^2+v^2]^{-s+n/2}$$

$$=\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\binom{-s+n/2}{k}v^{2k}(m+u)^{-2s+n-2k}$$

$$=\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(s-n/2+k)}{k!\Gamma(s-n/2)}v^{2k}$$

$$\times\sum_{r=0}^{\infty}\binom{-2s+n-2k}{r}u^r m^{-2s+n-2k-r}$$

$$=\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{k+r}\frac{\Gamma(s-n/2+k)}{k!\Gamma(s-n/2)}$$

$$\times\frac{\Gamma(2s-n+2k+r)}{r!\Gamma(2s-n+2k)}u^r v^{2k}\zeta(2s-n+2k+r)$$

$$\equiv\sum_{k,r\geqslant 0}C(s-n/2,k,r)u^r v^{2k}\zeta(2s-n+2k+r). \tag{2.7}$$

The last line defines the coefficients $C(s-n/2,k,r)$, some of whose properties are summarised in the appendix. To reach the final line we have commuted Σ_m through

Σ_k and Σ_r, and then evaluated Σ_m in terms of the Riemann ζ function

$$\zeta(s) \equiv \sum_{m=1}^{\infty} m^{-s} \qquad \mathrm{Re}\, s > 1. \tag{2.8}$$

That one may freely perform these commutations is rather easy to see. Assume that Re s is sufficiently large (i.e. Re $s > (n+1)/2$) that the series defining $\zeta(2s-n+2k+r)$ is absolutely convergent. Choose u and v small enough that both binomial series in these variables are absolutely convergent. Then all sums can be freely commuted, and (2.7) (the final equality) is the result. This formula invoves only functions which are known throughout the s plane. Hence it can be used for (or it automatically provides) the continuation of $Z(s-n/2, u, v)$ throughout the s plane. So now we know $Z(s-n/2, u, v)$ as a meromorphic function of s. This is generally what the Taylor series expansion of a generalised ζ function is able to provide.

For the fermion sum in (2.5) we have the power series expansion

$$\sum_{m=-\infty}^{\infty} [(m+\tfrac{1}{2}-u)^2+v^2]^{-s+n/2} = Z(s-n/2, u-\tfrac{1}{2}, v) + Z(s-n/2, -u-\tfrac{1}{2}, v) \tag{2.9}$$

where Z is the function in (2.7). For comparison with [9] it will be convenient to use the modified power series expansion

$$Z(s-n/2, u-\tfrac{1}{2}, v) = \sum_{k,r\geqslant 0} C(s-n/2, k, r) u^r v^{2k} \zeta(2s-n+2k+r, \tfrac{1}{2}) \tag{2.10}$$

with $C(s-n/2, k, r)$ as before and

$$\zeta(s, \tfrac{1}{2}) = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})^{-s} = (2^s-1)\zeta(s). \tag{2.11}$$

Comparison with (2.7) shows that the only difference between $Z(s-n/2, u, v)$ and $Z(s-n/2, u-\tfrac{1}{2}, v)$ is that $\zeta(s)$ in the Taylor series gets changed to $\zeta(s, \tfrac{1}{2})$.

From now on we assume that n is an odd integer ($n+1=2N$). The case n = even integer is significantly different.

2.2. *Boson gas thermodynamic potential*

To calculate Ω_{B}, we first note that, for infinitesimal $s=\varepsilon$,

$$Z_{\mathrm{B}}(\varepsilon) = VT^n \pi^{n/2} \frac{\Gamma(\varepsilon-n/2)}{\Gamma(\varepsilon)} \left(\frac{\beta M e^{\gamma}}{2\pi}\right)^{2\varepsilon} \times \{(u^2+v^2)^{-\varepsilon+n/2} + Z(\varepsilon-n/2, u, v) + Z(\varepsilon-n/2, -u, v)\} \tag{2.12}$$

where due to the factor $1/\Gamma(\varepsilon)=\varepsilon$, only the singular terms in the curly bracket contribute. These terms come from the pole in the Riemann ζ function $\zeta(2s-n+2k+r)$ at $2k+r=n+1$ in (2.7); thus

$$Z(\varepsilon-n/2, u, v) = \left(\frac{1}{2\varepsilon}+\gamma\right) Z_{\mathrm{pole}}(v) + P_{1\mathrm{B}}(u, v) + \Gamma_{\mathrm{B}}(u, v) + \mathrm{O}(\varepsilon) \tag{2.13}$$

where (see the appendix)

$$\begin{aligned} Z_{\mathrm{pole}}(v) &\equiv \sum_{2k+r=n+1} C(-n/2, k, r) u^r v^{2k} \\ &= C(-n/2, (n+1)/2, 0) v^{n+1} \\ &= (-1)^{(n+1)/2)} \frac{\sqrt{\pi} v^{n+1}}{[(n+1)/2)]!\Gamma(-n/2)} \end{aligned} \tag{2.14}$$

$$P_{1B}(u, v) \equiv \sum_{2k+r<n+1} C(-n/2, k, r) u^r v^{2k} \zeta(-n+2k+r) \tag{2.15}$$

$$\Gamma_B(u, v) \equiv \sum_{2k+r>n+1} C(-n/2, k, r) u^r v^{2k} \zeta(-n+2k+r). \tag{2.16}$$

In (2.13) we have used $\zeta(1+2\varepsilon) = 1/2\varepsilon + \gamma + O(\varepsilon)$. Note that all of the coefficients $C(-n/2, k, r)$ are finite. As expected, there is no singularity in $Z_B(s)$ at $s=0$,

$$Z_B(0) = VT^n \pi^{n/2} \Gamma(-n/2) Z_{\text{pole}}(v). \tag{2.17}$$

For the calculation of $Z'_B(0)$ we need

$$\frac{d}{ds} Z(\varepsilon - n/2, u, v) = -\frac{1}{2\varepsilon^2} Z_{\text{pole}} + \frac{1}{2\varepsilon} \{Z_{\text{pole}}[\psi(\tfrac{1}{2}) - \psi(-n/2)] + P_2(u, v)\} + \text{regular terms} \tag{2.18}$$

where

$$P_2(u, v) \equiv \sum_{\substack{2k+r=n+1 \\ k\geq 0, r\geq 1}} \frac{d}{ds} C(-n/2, k, r) u^r v^{2k}. \tag{2.19}$$

The regular terms in (2.18) do not contribute to $Z'_B(0)$. Now we know all terms in the derivative of the ζ function

$$\begin{aligned} Z'_B(\varepsilon) = VT^n \pi^{n/2} \frac{\Gamma(\varepsilon - n/2)}{\Gamma(\varepsilon)} \left(\frac{\beta M e^C}{2\pi}\right)^{2\varepsilon} & \\ \times \Bigg[[\psi(\varepsilon - n/2) - \psi(\varepsilon) + 2C + 2\ln(\beta M/2\pi)] & \\ \times [(u^2+v^2)^{-\varepsilon+n/2} + Z(\varepsilon - n/2, u, v) + Z(\varepsilon - n/2, -u, v)] & \\ + \Bigg(-(u^2+v^2)^{-\varepsilon+n/2} \ln(u^2+v^2) + \frac{d}{ds} Z(\varepsilon - n/2, u, v) & \\ + \frac{d}{ds} Z(\varepsilon - n/2, -u, v) \Bigg) \Bigg] & \end{aligned} \tag{2.20}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ and $\psi(\varepsilon) = -1/\varepsilon - \gamma + O(\varepsilon)$. Again the overall factor $1/\Gamma(\varepsilon)$ singles out the $1/\varepsilon$ terms inside the curly bracket. There are two $1/\varepsilon^2$ terms inside the bracket which must and do cancel. The result for Ω_B is

$$\begin{aligned} \Omega_B(M, \mu) = -VT^{n+1} \pi^{n/2} \Gamma(-n/2) \{ & (u^2+v^2)^{n/2} + \tilde{P}_2(u, v) + 2\tilde{P}_{1B}(u, v) + 2\tilde{\Gamma}_B(u, v) \\ & + Z_{\text{pole}}(v)[\psi(\tfrac{1}{2}) + 3\gamma + 2C + 2\ln(\beta M/2\pi)]\} \end{aligned} \tag{2.21}$$

where the tilde means symmetrisation in $u \leftrightarrow -u$;

$$\begin{aligned} \tilde{P}_{1B}(u, v) = \sum_{\substack{2k+2m<n+1 \\ k,m\geq 0}} (-1)^k \frac{\Gamma(-n/2+k)}{\Gamma(-n/2)} \frac{(n-2k)!}{k!(2m)!(n-2k-2m)!} & \\ \times u^{2m} v^{2k} \zeta(-n+2k+2m) & \end{aligned} \tag{2.22}$$

$$\tilde{P}_2(u, v) = \sum_{\substack{2k+2m=n+1 \\ k\geq 0, m\geq 1}} (-1)^{k+1} \frac{\Gamma(-n/2+k)}{\Gamma(-n/2)} \frac{u^{2m} v^{2k}}{k!m} \tag{2.23}$$

$$\tilde{\Gamma}_{\mathrm{B}}(u, v)=\sum_{\substack{2k+2m>n+1\\ k\geqslant\frac{1}{2}(n+1)\\ m\geqslant 0}}(-1)^k\frac{\Gamma(-n/2+k)}{\Gamma(-n/2)}\frac{(2k-n-1+2m)!}{(2k-n-1)!}\times\frac{u^{2m}v^{2k}}{k!(2m)!}\zeta(-n+2k+2m). \tag{2.24}$$

We now have to show that (2.21) agrees with previously derived results for Ω_{B}.

(1) The term containing $(u^2+v^2)^{n/2}$ is the one arising from series commutation in the other ζ-function approach [9-11]. We see that this term—an obvious one in the present derivation—is (with $n+1=2N$)

$$\begin{aligned}\Delta\Omega_{\mathrm{B}}&=-VT^{n+1}\pi^{n/2}\Gamma(-n/2)(u^2+v^2)^{n/2}\\&=(-1)^{N+1}\frac{2VT\pi^{N+3/2}}{(2\pi)^{2N}\Gamma(N+\frac{1}{2})}(M^2-\mu^2)^{N-1/2}\end{aligned} \tag{2.25}$$

in agreement with [6, 10]. We remind the reader that this term enforces the constraint $M^2-\mu^2\geqslant 0$ on the bosonic chemical potential [4].

(2) Another easily distinguished part of the formula is the contribution from the square bracket in (2.21). Again setting $n+1=2N$ ($N=s$ in the notation of [9]) we have for this contribution

$$\begin{aligned}&-VT^{n+1}\pi^{n/2}\Gamma(-n/2)Z_{\mathrm{pole}}(v)[\psi(\tfrac{1}{2})+3\gamma+2\ln(\beta M/2\pi)+2C]\\&\quad=(-1)^{N+1}\frac{2V\pi^N}{N!}\left(\frac{M}{2\pi}\right)^{2N}\left(\ln\frac{\beta M}{4\pi}+\gamma+C\right).\end{aligned} \tag{2.26}$$

Comparing this with (5.2) of [9] (note there is an overall factor π^{-N} missing in that formula) we find complete agreement, *if* the constant C has the value

$$C=-\tfrac{1}{2}\sum_{n=1}^{N}\frac{1}{n}. \tag{2.27}$$

Within the context of the present derivation, C appears to be arbitrary. Recall that this constant entered via the normalisation in (2.5). Because $Z_{\mathrm{B}}(0)\neq 0$ the determinant must depend on C, but there is no obvious way to pin down the value of C other than by the argument just given.

(3) Continuing our comparison of (2.21) with results derived previously, we next identify the polynomial contribution to Ω_{B}—called $P_{\mathrm{B}}(M, \mu)$ in [9]—which here comes from $\tilde{P}_2+2\tilde{P}_1$. The precise connection is

$$\tilde{P}_2(u, v)+2\tilde{P}_{1\mathrm{B}}(u, v)=(-1)^N\frac{4\Gamma(N+\frac{1}{2})}{\pi^{2N+1/2}}P_{\mathrm{B}}(M, \mu) \tag{2.28}$$

where again $n+1=2N$ and

$$P_{\mathrm{B}}(M, \mu)=\tfrac{1}{2}\sum_{c=0}^{N}\frac{(\beta\mu)^{2c}}{(2c)!}\sum_{a=0}^{N-1}\frac{\Gamma(N-a)}{a!}(-1)^a\left(\frac{\beta M}{2}\right)^{2a}\zeta(2N-2a-2c).$$

One finds in the physical case of three spatial dimensions ($n=3$)

$$\begin{aligned}\tilde{P}_{1\mathrm{B}}(u, v)&=\tfrac{1}{120}-\tfrac{1}{8}(2u^2+v^2)\\\tilde{P}_2(u, v)&=-\tfrac{1}{2}(u^4+3u^2v^2)\end{aligned} \tag{2.29}$$

and quickly verifies that the correct polynomial contribution to Ω_{B} is obtained.

(4) All that remains is the infinite series contribution to Ω_B—called $S_B(M, \mu)$ in [9]. This contribution comes from $\tilde{\Gamma}_B(u, v)$:

$$\tilde{\Gamma}_B(u, v) = (-1)^N \frac{2\Gamma(N+\frac{1}{2})}{\pi^{2N+1/2}} S_B(M, \mu) \tag{2.30}$$

where

$$S_B(M, \mu) = (-1)^N \left(\frac{\beta M}{2}\right)^{2N} \sum_{\substack{b,c\geq 0 \\ b+c>0}} (-1)^{b+c} \left(\frac{\beta M}{4\pi}\right)^{2b} \left(\frac{\beta\mu}{2\pi}\right)^{2c} \times \frac{(2b+2c)!}{b!(N+b)!2(2c)!} \zeta(1+2b+2c).$$

Equation (2.30) is actually quite easy to verify.

2.3. Fermi gas thermodynamic potential

With minor changes, the preceding calculation gives us Ω_F as well. Because

$$\zeta(1+2\varepsilon, a) = \frac{1}{2\varepsilon} - \psi(a) + O(\varepsilon)$$

equation (2.13) becomes now

$$Z(\varepsilon - n/2, u-\tfrac{1}{2}, v) = \left(\frac{1}{2\varepsilon} - \psi(\tfrac{1}{2})\right) Z_{\text{pole}}(v) + P_{1F}(u, v) + \Gamma_F(u, v) + O(\varepsilon) \tag{2.31}$$

where $\psi(\frac{1}{2}) = -\gamma - 2\ln 2$ and $Z_{\text{pole}}(v)$ is the same as before. $P_{1F}(u, v)$ and $\Gamma_F(u, v)$ are defined by (2.15) and (2.16) with $\zeta(s, \frac{1}{2})$ replacing $\zeta(s)$ in these formulae. Note that (2.18) can be used unchanged.

The final result for the thermodynamic potential Ω_F is

$$\Omega_F(M, \mu) = V(d/2)T^{n+1}\pi^{n/2}\Gamma(-n/2) \times \{2\tilde{P}_{1F}(u, v) + \tilde{P}_2(u, v) + 2\tilde{\Gamma}_F(u, v) + 4\ln 2\, Z_{\text{pole}}(v) + Z_{\text{pole}}(v)[2\ln(\beta M/4\pi) + 2\gamma + 2C)\}. \tag{2.32}$$

Let us compare this expression term by term with the previous result.

(1) The contribution from the square bracket in (2.32) is (choosing $n+1 = 2N$)

$$\tfrac{1}{2} dVT^{n+1}\pi^{n/2}\Gamma(-n/2)2Z_{\text{pole}}(v)[\ln(\beta M/4\pi) + \gamma + C] = (-1)^N dVT^{2N}(\beta M/2)^{2N}(N!\pi^N)^{-1}[\ln(\beta M/4\pi) + \gamma + C]. \tag{2.33}$$

With the same value (2.27) for C, this agrees with equation (4.11) of [9]. (The latter equation has (i) a factor π^{-N} missing and (ii) a minus sign missing before its square bracket.)

(2) The polynomial contribution to Ω_F is

$$\tfrac{1}{2} dVT^{n+1}\pi^{n/2}\Gamma(-n/2)[2\tilde{P}_{1F}(u, v) + \tilde{P}_2(u, v)] = -VT^{2N} 2dP_F(M, \mu). \tag{2.34}$$

In four spacetime dimensions

$$\tilde{P}_{1F}(u, v) = -\tfrac{7}{8}\tfrac{1}{120} + \tfrac{1}{8}u^2 + \tfrac{1}{16}v^2 \tag{2.35}$$

(for $\tilde{P}_2$ see (2.29)) and one easily verifies that (2.34) correctly gives $P_F(M, \mu)$.

(3) The infinite series contribution to Ω_F is

$$V(d/2)T^{n+1}\pi^{n/2}\Gamma(-n/2)[2\tilde{\Gamma}_F(u,v)+4\ln 2\, Z_{\text{pole}}(v)] = -VT^{2N}(2d/\pi^N)S_F(M,\mu) \tag{2.36}$$

where in agreement with [9]

$$S_F(M,\mu)+(-1)^N\frac{\ln 2}{N!}\left(\frac{\beta M}{2}\right)^{2N} = (-1)^N \sum_{\substack{b,c\geq 0\\ b+c>0}} (-1)^{b+c}\left(\frac{\beta M}{4\pi}\right)^{2b}\left(\frac{\beta\mu}{2\pi}\right)^{2c}\left(\frac{\beta M}{2}\right)^{2N} \times\frac{(2b+2c)!}{b!(N+b)!2(2c)!}(1-2^{1+2b+2c})\zeta(1+2b+2c). \tag{2.37}$$

Equation (2.37) is easily verified.

3. Thermodynamic potentials on $M\times E^n$

The calculation of thermodynamic potentials on spacetime toroidal manifolds $T^N\times E^n$ can be done very much like the preceding calculations on $S^1\times E^n$. Here we indicate the main steps, without evaluation of the toroidal (Epstein) ζ functions involved. (Some related discussions are [19-21].) The same approach is also well suited for dealing with other manifolds, e.g. $S^N\times E^n$, as we briefly describe at the end of the section.

The ζ function for the vacuum scalar loop on $T^N\times E^n$ is

$$Z_B(s)(M\mathrm{e}^c)^{-2s} \equiv \frac{V}{(2\pi)^n}\int \mathrm{d}^n k \sum_{m_i}[(\omega^+_{m_1}-B_1)^2+\ldots+(\omega^+_{m_N}-B_N)^2+k^2+M^2]^{-s} \tag{3.1}$$

where m_i is any integer, $\omega^+_{m_i}=2\pi m_i/\beta_i$, β_i is the circumference of the ith circle in T^N, and $i=1,2,\ldots,N$. B_i is a constant Abelian gauge potential which enters non-trivially (as a kind of generalised complex chemical potential). Just as in (2.5), the momentum integral leads to

$$Z_B(s)=V\beta_1^{-n}\pi^{n/2}\left(\frac{\beta_1 M}{2\pi}\mathrm{e}^c\right)^{2s}\frac{\Gamma(s-n/2)}{\Gamma(s)}\zeta_N(s) \tag{3.2}$$

$$\zeta_N(s)\equiv\sum_{m_i}[(m_1-u_1)^2+(a_2m_2-u_2)^2+\ldots+(a_Nm_N-u_N)^2+v^2]^{-s+n/2} \tag{3.3}$$

where $u_i=\beta_1 B_i/2\pi$, $v=\beta_1 M/2\pi$, $a_i=\beta_1/\beta_i$ and β_1 is playing the role of inverse temperature in the notation. What remains is the evaluation of the ζ function (3.3)—by far the most difficult part of the problem. Once this has been accomplished, the evaluation of the thermodynamic potential (2.1) is straightforward.

In the limit of $M=0$, $a_i=1$ and $u_i=0$ or $\frac{1}{2}$, the ζ function (3.3) is an Epstein ζ function [22] which in many cases [23] has a known expression in terms of Riemann-type ζ functions. $Z_B(s)$ is then known, and Ω_B can be obtained immediately.

More general cases involving one or more of $M>0$, $a_i\neq 1$ and arbitrary u_i have received little attention because the ζ function (3.3) is unknown for these cases. To obtain a solution like the one in § 2 for $N=1$ we clearly need the power series expansion of $\zeta_N(s)$ in v^2 and u_i. Known results on Epstein ζ functions do not provide these power series. Recently [24], a method for finding such expansions has been developed.

We hope to provide a full calculation of the power series of $Z_B(s)$, and therefore Ω_B, in the near future. This is too lengthy to discuss in more detail here.

For manifolds $M \times E^n$ with non-toroidal compactification, one needs to know the eigenvalues λ_m of the Laplacian $\square_N = \partial_1^2 + \ldots + \partial_N^2$ on M, and the multiplicities g_m of these eigenvalues. Then the ζ function for the scalar vacuum loop is defined by

$$Z_B(s)(Me^c)^{-2s} \equiv \frac{V}{(2\pi)^n} \int d^n k \sum_m g_m(\lambda_m + \boldsymbol{k}^2 + M^2)^{-s}. \tag{3.4}$$

The integral is readily done to express $Z_B(s)$ in a form like (3.2), involving the ζ function

$$\sum_m g_m(\lambda_m + M^2)^{-s+n/2}. \tag{3.5}$$

By dimensional reasoning $\lambda_m = h_m/a^2$, where $a \sim$ length and h_m is typically a polynomial in m, as is g_m. Thus equation (3.5) defines a ζ function of a type encountered in many other problems, which can be calculated. This will yield $Z_B(s)$ and Ω_B.

Acknowledgments

This article is based on work done at the University of Salamanca. The author thanks Jose Cervero and all other members of the theory group there for their kind hospitality. Support from the CAICYT, the Comite Conjunto Hispano-Norteamericano Para la Cooperacion Cientifica y Tecnologica, and the Penn State FSSF is gratefully acknowledged.

Appendix

We shall need some properties of the constants

$$C(s-n/2, k, r) \equiv (-1)^{k+r} \frac{\Gamma(s-n/2+k)\Gamma(2s-n+2k+r)}{k!\,r!\,\Gamma(s-n/2)\Gamma(2s-n+2k)} \tag{A1}$$

defined in (2.7) of the text. These properties are

$$C(-n/2, k, r) = 0 \qquad 2k+r > n+1 \qquad 2k \leqslant n \tag{A2}$$

$$C(-n/2, k, r) = (-1)^k \frac{\Gamma(-n/2+k)(n-2k)!}{\Gamma(-n/2)k!\,r!\,(n-2k-r)!} \qquad 2k+r < n+1 \tag{A3}$$

$$\frac{d}{ds} C(-n/2, k, 0) = C(-n/2, k, 0) \times [\psi(-n/2+k) - \psi(-n/2)] \tag{A4}$$

$$\frac{d}{ds} C(-n/2, k, r) = (-1)^{k+1} \frac{2}{r} \frac{\Gamma(-n/2+k)}{\Gamma(-n/2)k!}. \qquad 2k+r = n+1 \qquad r \geqslant 1 \tag{A5}$$

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